On classes of non-Gaussian asymptotic minimizers in entropic uncertainty principles

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Abstract

In this paper we revisit the Bialynicki-Birula & Mycielski uncertainty principle [1] and its cases of equality. This Shannon entropic version of the well-known Heisenberg uncertainty principle can be used when dealing with variables that admit no variance. In this paper, we extend this uncertainty principle to Rényi entropies. We recall that in both Shannon and Rényi cases, and for a given dimension n, the only case of equality occurs for Gaussian random vectors. We show that as n grows, however, the bound is also asymptotically attained in the cases of n-dimensional Student-t and Student-r distributions. A complete analytical study is performed in a special case of a Student-t distribution. We also show numerically that this effect exists for the particular case of a n-dimensional Cauchy variable, whatever the Rényi entropy considered, extending the results of Abe [2] and illustrating the analytical asymptotic study of the student-t case. In the Student-r case, we show numerically that the same behavior occurs for uniformly distributed vectors. These particular cases and other ones investigated in this paper are interesting since they show that this asymptotic behavior cannot be considered as a "Gaussianization" of the vector when the dimension increases.

Key words: Entropic uncertainty relation, Rényi/Shannon entropy, multivariate Student-t and Student-r distributions.

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1 Introduction

Let us consider an n-dimensional wave packet $\Psi_n(x)$ and denote by $\widehat{\Psi}_n(u) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \Psi_n(x) \mathrm{e}^{-iu^t x} \, \mathrm{d}x$ its Fourier transform (consider for example the position of a particle and its momentum). In the following, we will denote by X_n a zero-mean random vector with probability density function (pdf) $f_n(x) = |\Psi_n(x)|^2$ and by \widetilde{X}_n a random vector with pdf $\widetilde{f}_n(x) = |\widehat{\Psi}(x)|^2$ (by Parseval's relation, this is a pdf). Vectors X_n and \widetilde{X}_n are called conjugated. The well-known Heisenberg uncertainty principle (H.U.P.) relates the "information" available in two conjugated random vectors, stating that the product of their variances is larger than a given bound, namely

$$\frac{\left(E\left[X_n^t X_n\right] E\left[\widetilde{X}_n^t \widetilde{X}_n\right]\right)^{\frac{1}{2}}}{n} \ge \frac{1}{2} \tag{1}$$

(see also [3] for a matrix-variate extension of this result). The H.U.P. is important in physics since it expresses the impossibility of an arbitrarily accurate preparation of both position and momentum of a particle. This inequality finds also application in areas of signal processing such as time-frequency analysis [4,5] where it is known as the Heisenberg-Gabor inequality. However, the H.U.P. has a meaning only if the quantities in balance exist.

To bypass this restriction, Bialynicki-Birula & Mycielski showed in 1975 [1] that the H.U.P. can be extended to information theoretic measures: more precisely, they showed that the sum of the Shannon entropy rates of X_n and of X_n verifies the Bialynicki-Birula & Mycielski inequality (B.B.M.I.)

$$\frac{H(X_n) + H(\widetilde{X}_n)}{n} \ge 1 + \log \pi \tag{2}$$

where the Shannon entropy is $H(X_n) = -\int f_n \log f_n$ (and likewise for \widetilde{X}_n). The B.B.M.I. can be expressed equivalently via the entropy power $N(X_n) = \frac{1}{2\pi e} \exp\left(\frac{2}{n}H(X_n)\right)$ (as given in [3]) by

$$\left(N(X_n)N(\widetilde{X}_n)\right)^{\frac{1}{2}} \ge \frac{1}{2}.\tag{3}$$

The B.B.M.I. (2) is stronger than the H.U.P. (1): as shown in [1], it can be applied also to variables with infinite variance, provided that their Shannon entropy exists: it is the case for a Cauchy pdf $f_n(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}(1+x^tx)^{-\frac{n+1}{2}}$, for example. Moreover, it is shown in [1] that B.B.M.I. (2) implies H.U.P. (1) when

dealing with variables that admit a variance 1. As inequality (1), the B.B.M.I. finds applications in signal processing (see [4,5] and references therein). In physics, maximization without constraint of the sum of the entropies that appear in the B.B.M.I. has been suggested [6] as an interesting counterpart to the classical "maximum entropy under constraint" approach for the derivation of the wave functions associated with atomic systems. As for the Heisenberg inequality, the lower bound in (2) is attained only for Gaussian wave packets. But in their paper [2], Abe et al. showed that this bound is also asymptotically reached by n-dimensional Cauchy vectors when n increases. In this paper, we will extend this observation by exhibiting other families of distributions that show the same asymptotical behavior. Moreover, we will focus on the possible interpretations of this behavior, namely a "Gaussianization effect": one natural interpretation is that these distributions get closer to the Gaussian distribution - in some sense to be determined - as the dimension increases; this interpretation will be proved erroneous by providing some cases in which a distribution reaches asymptotically the bound of the B.B.M.I. but keeps at non-zero (even infinite) distance of any Gaussian distribution. As we will explain in the conclusion, the effect observed is mainly due to the normalization 1/n, which may be too strong in the considered non-iid context.

This paper is organized as follows:

- In the first part, we come back to the Bialynicki-Birula & Mycielski uncertainty relation, we reformulate it and we generalize it to the Rényi entropies. We then give the expression of the entropies rates under consideration when dealing with elliptical random vectors.
- We then address the study of the asymptotic cases of equality in the generalized B.B.M.I., particularizing to the family of n-dimensional Student-t variables with m degrees of freedom. For this class of variables, we provide an upperbound of the sum of the entropy rates in balance which permits to evaluate the asymptotic behavior of this quantity. We then perform the complete analytical study in the particular case m = n + 2, confirming that the lower bound of the uncertainty relation is attained asymptotically as n increases. In a second illustration, we revisit the Cauchy case (m = 1) as studied by Abe in the context of the Rényi formulation of the uncertainty relation, and we show that the effect observed by Abe remains for any "admissible" Rényi entropy.

Secondly, we explore what happens in the n-dimensional Student-r class of variables with m degrees of freedom. We present the case m=n, corresponding to the uniform distribution in the n-dimensional sphere, as well

¹ Since under covariance constraint N is maximum in the Gaussian context, $N(X_n) \leq N(G_n) = n\sigma^2$ where G_n is Gaussian with the same covariance than X_n , and similarly for \widetilde{X}_n : the product of the variances is higher than the product of the entropy power, implying the H.U.P.

as some other cases.

• Finally, we provide some clarifications about this asymptotic behavior, explaining why a Gaussianization effect, measured in the distribution or in the information divergence sense, is not necessary to reach asymptotically equality in the B.B.M.I.

2 The Rényi entropy uncertainty relation

The proof of the B.B.M.I. (2) is based on the Beckner inequality relating the norms of any (wave) function Ψ_n of $L^p(\mathbb{R}^n)$ and of its Fourier transform $\widehat{\Psi}_n$, *i.e.*

$$\|\widehat{\Psi}_n\|_q \le (C_{p,q})^n \|\Psi_n\|_p \tag{4}$$

where p and q are conjugated, i.e. $\frac{1}{p} + \frac{1}{q} = 1$, where $p \in]1;2]$ and where $C_{p,q}$ is the Babenko constant expressed as $C_{p,q} = \left(\frac{2\pi}{p}\right)^{-\frac{1}{2p}} \left(\frac{2\pi}{q}\right)^{\frac{1}{2q}}$ [7,8]. In [1], the authors consider the positive function $W(q) = (C_{p,q})^n \|\Psi_n\|_p - \|\widehat{\Psi}_n\|_q$ defined for $q \geq 2$. Since W(2) = 0 by Parseval's identity and since W(q) is positive, the derivative of W(q) in q = 2 is positive as well, which leads to inequality (2).

The B.B.M.I. (2) can be extended to other measures of information such as the Rényi entropies that include the Shannon entropy as a special case. The Rényi entropy with parameter λ is defined as

$$H_{\lambda}(X_n) = \frac{1}{1-\lambda} \log\left(\int f_n^{\lambda}\right) \tag{5}$$

for $\lambda \neq 1$ [9]. When λ tends to 1, by l'Hospital's rule, H_{λ} converges to the Shannon entropy that will thus be denoted by continuity $H_1 = H$. The Rényi entropy is widely used, not only in physics (e.g. statistical mechanics, physics of turbulence, cosmology, see [10,11,12] and references therein), but in various other areas such as in signal processing (time scale analysis, decision problems, machine learning, see [4,13,14,15] and references therein), or image processing (image matching, image registration see [16,17] and references therein).

Theorem 1 With $\frac{1}{p} + \frac{1}{q} = 1$ and for any p > 1, the B.B.M.I. writes in terms of Rényi entropy as

$$\frac{H_{\frac{p}{2}}(X_n) + H_{\frac{q}{2}}(\widetilde{X}_n)}{n} \ge \log(2\pi) + \frac{\log p}{p-2} + \frac{\log q}{q-2}.$$
 (6)

Proof. It is straightforward that

$$\log \|\Psi_n\|_p = \log \|f_n^{1/2}\|_p = \frac{1}{p} \log \left(\int f_n^{p/2} \right)$$
$$= \frac{2-p}{2p} H_{\frac{p}{2}}(X_n).$$

Hence, taking the logarithm of both sides of (4), using the conjugation relation $\frac{1}{p} + \frac{1}{q} = 1$ and 1 leads to

$$\frac{2-p}{2p} H_{\frac{p}{2}}(X_n) - \frac{2-q}{2q} H_{\frac{q}{2}}(\widetilde{X}_n) \ge n \left(\frac{1}{2p} - \frac{1}{2q}\right) \log(2\pi) + \frac{n}{2p} \log p - \frac{n}{2q} \log q.$$

But p and q are conjugated so that $\frac{2-q}{2\,q} = \frac{p-2}{2\,p}$. Hence, since $2-p \geq 0$, multiplying each side by $\frac{2\,p}{n\,(2-p)}$ one finally obtains uncertainty relation (6).

Since the pdfs of \widetilde{X}_n and of X_n coincide, X_n and \widetilde{X}_n have a symmetrical role in (6) and can then be exchanged: as a consequence inequality (6) holds for any p > 1, provided that the entropies in balance exist.

By taking the limit $p \to 2$ in (6), the B.B.M.I. (2) is recovered, proving that it is a particular case of (6). We notice that a similar generalization exists for the Tsallis entropy [18,19,20], which is widely used in statistical physics and related to the Rényi entropy by an invertible transformation [10,11,12,21,22]. In this paper we will focus on the Rényi entropy since the resulting form of the uncertainty relation is very similar to the B.B.M.I. Furthermore, the quantity $\frac{1}{n}H_{\lambda}(X_n)$, also known as the entropy rate (*i.e.* entropy per sample), is very often considered in the information theoretic context [13].

Theorem 2 For a given n, case of equality in (2) or (6) is reached if and only if X_n and \widetilde{X}_n are Gaussian random vectors.

Proof. It is straightforward to check that Gaussian vectors X_n and X_n reach equality in either inequality (2) or (6) by plugging the Gaussian pdfs. Conversely, for $p \neq 2$, since only Gaussian waves functions achieve equality in (4) as it is proved in [23], the Gaussian waves are the *only* wave packets which achieve equality in (6). The Shannon case p=2 is more subtle, but it has been recently proved that equality is reached in inequality (2) *only* in the Gaussian case [24].

However, in [2], Abe showed in the case of Shannon entropies that n-variate Cauchy vectors reach equality in (2) asymptotically with n. We will show in the next part that this result extends in fact to inequality (6) for any value

p/2 of the entropy index for which the entropy exists. Furthermore we will show that the Cauchy case is not the only one exhibiting this behavior.

3 Asymptotic cases of equality

As previously stated, although equality is achieved in (6) only by Gaussian wave packets, n-dimensional Cauchy wave packets reach equality asymptotically with the dimension n. An important question is to understand what are the ingredients that lead to this asymptotic behavior.

We first remark that in the Cauchy case presented by Abe and in the cases studied below, the components of the random vector are dependent: in other words the wave function Ψ_n is not separable. This condition is clearly required since, dealing with independent identically distributed (i.i.d.) components (separable wave function Ψ_n), the entropy rate coincides with the entropy of a single component: $H_{\lambda}(X_n) = nH_{\lambda}(X)$ where X is any component of X_n . But if X_n is i.i.d., the conjugated vector \widetilde{X}_n is i.i.d. as well since the Fourier transform of a separable function is separable. Hence with obvious notations $H_{\lambda}(\widetilde{X}_n) = nH_{\lambda}(\widetilde{X})$ and $\frac{H_{\lambda}(X_n) + H_{\lambda}(\widetilde{X}_n)}{n} = H_{\lambda}(X) + H_{\lambda}(\widetilde{X})$: as a consequence, no asymptotic effect can appear in the i.i.d. setup.

Furthermore, for any invertible matrix M, the Rényi λ -entropy of $Y_n = MX_n$ is expressed as $H_{\lambda}(Y_n) = \log |M| + H_{\lambda}(X_n)$ and since $\widetilde{Y}_n = M^{-t}\widetilde{X}_n$, the sum of the entropy rates is (matrix) scale invariant: it is thus impossible to reach equality by introducing such simple correlation between the components of the random vectors.

Except for these basic requirements, the answer remains open as far as we know. The study of the following cases, showing very different behaviors, attempts to give some elements of answer.

3.1 Derivation of the entropy rates in the elliptic case

We concentrate in the following on the case of elliptical random vectors. A vector X_n is called elliptical if its pdf f_n is a single-valued function of a quadratic form [25]

$$f_n(x) = |\Sigma_n^{-1}|^{-\frac{1}{2}} d_n \left(\left(x^t \Sigma_n^{-1} x \right)^{\frac{1}{2}} \right)$$
 (7)

for some function d_n and where Σ_n is a positive definite symmetric matrix. Σ_n is called characteristic matrix. In other words, the random vector $\Sigma_n^{-1/2}X_n$ is isotropic. Due to the matrix scale invariance of the studied entropic inequalities evoked above, we will consider without loss of generality in the following that Σ_n is proportional to the identity matrix I_n . Otherwise, except in (6) where there is no influence, X_n and \widetilde{X}_n must be understood as $\Sigma_n^{-1/2}X_n$ and $\Sigma_n^{1/2}\widetilde{X}_n$ respectively.

Theorem 3 If X_n is elliptical as in (7) (with $\Sigma_n = I_n$), then the sum of the entropy rates of X_n and \widetilde{X}_n is

$$U_{p}(X_{n}) = \frac{H_{\frac{p}{2}}(X_{n}) + H_{\frac{q}{2}}(\widetilde{X}_{n})}{n}$$

$$= \frac{2}{n} \log \left(\frac{2\pi^{n/2}}{\Gamma(n/2)} \right)$$

$$+ \frac{2}{n(2-p)} \log \int_{0}^{+\infty} r^{\frac{(n-1)(2-p)}{2}} D_{n}(r)^{\frac{p}{2}} dr$$

$$+ \frac{2}{n(2-q)} \log \int_{0}^{+\infty} r^{\frac{(n-1)(2-q)}{2}} E_{n}(r)^{\frac{q}{2}} dr$$
(8)

for $p \neq 2$ and

$$U_{2}(X_{n}) = \frac{H(X_{n}) + H(\widetilde{X}_{n})}{n}$$

$$= \frac{2}{n} \log \left(\frac{2\pi^{n/2}}{\Gamma(n/2)}\right) + \frac{n-1}{n} \int_{0}^{+\infty} \log r \left(D_{n}(r) + E_{n}(r)\right) dr$$

$$-\frac{1}{n} \int_{0}^{+\infty} \left(D_{n}(r) \log(D_{n}(r)) + E_{n}(r) \log(E_{n}(r))\right) dr$$
(9)

for p = 2, where

$$\begin{cases}
D_n(r) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} r^{n-1} d_n(r) \\
E_n(r) = \left(\int_0^{+\infty} (\rho r)^{\frac{1}{2}} D_n(\rho)^{\frac{1}{2}} J_{\frac{n}{2}-1}(\rho r) d\rho \right)^2
\end{cases}$$
(10)

are the pdfs of the Euclidean norms $||X_n||$ and $||\widetilde{X}_n||$ respectively.

Proof. Using spherical coordinates, a simple computation shows that the pdf $D_n(r)$ of the Euclidean norm $||X_n||$ of X_n is

$$D_n(r) = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} r^{n-1} d_n(r) \tag{11}$$

if $r \ge 0$ and is zero otherwise (see also [26, eq. (7)]).

Let us show now that, X_n being elliptical, \widetilde{X}_n is elliptical: the pdf \widetilde{f}_n of the conjugate variable is given by

$$\widetilde{f}_n(u) = \left| (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f_n^{\frac{1}{2}}(x) e^{-\imath u^t x} dx \right|^2$$

$$= \left| (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} d_n^{\frac{1}{2}} \left((x^t x)^{\frac{1}{2}} \right) e^{-\imath u^t x} dx \right|^2$$

Applying [26, eq. (5)], we obtain

$$\widetilde{f}_n(u) = \left| \int_0^{+\infty} \rho^{\frac{n}{2}} (u^t u)^{-\frac{n-2}{4}} J_{\frac{n}{2}-1}(\rho(u^t u)^{\frac{1}{2}}) d_n^{\frac{1}{2}}(\rho) d\rho \right|^2$$

where J_{ν} is the Bessel function of the first kind and order ν . This result proves that \widetilde{X}_n is elliptical. From the above expression of \widetilde{f}_n and the forms (7)-(11), the pdf E_n of $\|\widetilde{X}_n\|$ writes then

$$E_n(r) = \left(\int_0^{+\infty} (\rho r)^{\frac{1}{2}} D_n^{\frac{1}{2}}(\rho) J_{\frac{n}{2}-1}(\rho r) d\rho\right)^2$$
(12)

for $r \geq 0$ and zero otherwise. We remark that E_n is the square of the $(\frac{n}{2} - 1)$ order Hankel transform of $D_n^{\frac{1}{2}}$ (see also [27,28]).

Now the λ -Rényi entropy of X_n is

$$H_{\lambda}(X_n) = \frac{1}{1-\lambda} \log \int_{\mathbb{R}^n} f_n^{\lambda}(x) dx$$
$$= \frac{1}{1-\lambda} \log \left(\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \int_0^{+\infty} r^{n-1} d_n^{\lambda}(r) dr \right)$$

the last equality being an application of [29, 4.642]. This yields the following expression for the λ -Rényi entropy of X_n ,

$$H_{\lambda}(X_n) = \log \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} + \frac{1}{1-\lambda} \log \int_{0}^{+\infty} r^{(n-1)(1-\lambda)} D_n^{\lambda}(r) dr$$
 (13)

The Shannon entropy of X_n can be deduced from (13) using L'Hospital's rule:

$$H(X_n) = \log \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} + (n-1) \int_0^{+\infty} D_n(r) \log r \, dr - \int_0^{+\infty} D_n(r) \log D_n(r) \, dr \quad (14)$$

We have shown that \widetilde{X}_n is again elliptical so that its λ -Rényi entropy is again expressed as (13) with D_n , the pdf of $||X_n||$, replaced by E_n , the pdf of $||\widetilde{X}_n||$, leading to results (8) and (9).

We have now all the material to investigate more deeply some special cases of asymptotic equalities in (6).

3.2 The general Student-t case

We consider in this subsection the case where X_n is distributed according to a Student-t law with m degrees of freedom, *i.e.* ²

$$f_n(x) = \frac{\Gamma\left(\frac{n+m}{2}\right)}{\pi^{\frac{n}{2}}\Gamma\left(\frac{m}{2}\right)} \left(1 + x^t x\right)^{-\frac{n+m}{2}} \tag{15}$$

where m > 0. When m = 1, X_n is the well-known multivariate Cauchy vector, as studied by Abe in [2]. The Student-t variables play an important role in statistics because of their power-law behavior and their simple analytic expression. Moreover, they maximize the Rényi/Tsallis entropy with parameter $\lambda = 1 - \frac{2}{n+m}$ under covariance constraint [12,21,22,31,32].

Theorem 4 If X_n is Student-t distributed, then the pdf of $||X_n||$ is

$$D_n(r) = \frac{2\Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)} \frac{r^{n-1}}{(1+r^2)^{\frac{m+n}{2}}}$$
(16)

 $[\]overline{}^2$ More rigorously the random variable $\sqrt{m}X_n$ is Student-t [30], but we have seen that the sum of the entropy rates is insensitive to any scaling factor.

while the pdf of $\|\widetilde{X}_n\|$ is

$$E_n(r) = \frac{2^{3 - \frac{n+m}{2}} \Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{m}{2}\right) \Gamma^2\left(\frac{m+n}{4}\right)} r^{\frac{n+m}{2} - 1} K_{\frac{n-m}{4}}^2(r)$$
(17)

where K_{ν} is the modified Bessel function of the third kind with order ν (see for example [29, 8.432]).

Moreover, if $p > \frac{2n}{n+m}$, the sum of the entropy rates is

$$U_{p}(X_{n}) = \log \pi + \frac{(4+q(4-n-m))\log 2}{2n(2-q)} + \frac{2}{n(2-p)} \left((p-1)\log \Gamma\left(\frac{n}{2}\right) + \log \Gamma\left(\frac{(n+m)p-2n}{4}\right) - \log \Gamma\left(\frac{(n+m)p}{4}\right) + p\log \Gamma\left(\frac{n+m}{4}\right) \right) + \frac{2}{n(2-q)}\log \int_{0}^{+\infty} r^{\frac{(m-n)q}{4}+n-1} K_{\frac{n-m}{4}}^{q}(r) dr$$
(18)

while, for p=2,

$$U_{2}(X_{n}) = \log \pi + \frac{n-2}{n} \log 2 + \frac{2}{n} \left(\log \Gamma \left(\frac{m}{2} \right) - \log \Gamma \left(\frac{n+m}{2} \right) + \log \Gamma \left(\frac{n+m}{4} \right) \right)$$

$$+ \frac{n-m}{4n} \left(\psi \left(\frac{n}{2} \right) + 2 \psi \left(\frac{n+m}{4} \right) \right) + \frac{m}{n} \psi \left(\frac{n+m}{2} \right) - \frac{n+3m}{4n} \psi \left(\frac{m}{2} \right)$$

$$- \frac{2^{3-\frac{n+m}{2}} \Gamma \left(\frac{n+m}{2} \right)}{n \Gamma \left(\frac{n}{2} \right) \Gamma \left(\frac{m+m}{4} \right)} \int_{0}^{+\infty} r^{\frac{n+m}{2}-1} K_{\frac{n-m}{4}}^{2}(r) \log K_{\frac{n-m}{4}}^{2}(r) dr$$

$$(19)$$

Proof. (16) is straightforward from (15), (7) and (11) while (17) is a consequence of (12) and [28, 9-28 (22)] (or [27, 8.5 (20)] or [29, 6.565–4]). Finally, plugging (16) and (17) into (8) and using [29, 8.380–3] and $q = \frac{p}{p-1}$, the sum of the entropy rates (18) for $p \neq 2$ follows. In the Shannon case, starting from (9), recognizing $\int_{0}^{+\infty} \frac{r^{n-1}}{(1+r^2)^{\frac{n+m}{2}}} dr \text{ as a beta integral and using [29, 6.576–4]}$ to evaluate $\int_{0}^{+\infty} r^{-\lambda} K_{\nu}^{2}(r) dr$, we notice that $h(r)^{\lambda} \log(h(r)) = \frac{\partial}{\partial \lambda} h(r)^{\lambda}$ (with h(r) = r and $h(r) = 1 + r^{2}$) to finally obtain (19).

Notice that both from [29, 8.380–3] to insure the existence of terms in (18) or from the asymptotics [30, 9.6.9 and 9.7.2] to insure the existence of the remaining integral, quantity $U_p(X_n)$ exists provided that

$$p > \frac{2n}{n+m} \tag{20}$$

Unfortunately, in the general case, neither (18) nor (19) can be further analytically developed: numerical integration is necessary for the evaluation of the remaining integral. We will see in section 3.2.1 that in the special case m = n + 2, the remaining integral can be fully developed and hence the investigation can be completely performed. However, an asymptotic result can be obtained for any positive value of m, as expressed by the following theorem.

Theorem 5 For any positive value of the degree of freedom, the following equivalent holds:

$$U_p(X_n) = \log(2\pi) + \frac{\log p}{p-2} + \frac{\log q}{q-2} + o(1) \qquad \text{for } p \neq 2$$
 (21)

$$U_2(X_n) = 1 + \log \pi + o(1) \tag{22}$$

Hence, the lower bound of (6) is asymptotically attained when $n \to \infty$.

Proof. Let us denote

$$I(\lambda, q) = \log \left(\int_{0}^{+\infty} r^{-1} \left(r^{\frac{m-n}{4} + \frac{n}{q}} K_{\frac{n-m}{4}}(r) \right)^{\lambda} dr \right)$$
 (23)

Schwarz inequality implies that $\frac{\partial^2 I}{\partial \lambda^2} \geq 0$, showing that function $I(\lambda,q)$ is convex against λ . As a consequence, for any $\lambda \in [1;2]$ we have $I(\lambda,q) = I((2-\lambda)\times 1 + (\lambda-1)\times 2, q) \leq (2-\lambda)I(1,q) + (\lambda-1)I(2,q)$. If $\lambda > 2$ this inequality is reversed. Since the remaining integral is I(q,q), for $q \neq 2$ we obtain the inequality

$$\frac{1}{2-q}\log\left(\int_{0}^{+\infty}r^{\frac{(m-n)q}{4}+n-1}K_{\frac{n-m}{4}}^{q}(r)\,\mathrm{d}r\right) \leq I(1,q) + \frac{q-1}{2-q}I(2,q) \tag{24}$$

Using [29, 6.561–16 and 6.576–4] to evaluate I(1,q) and I(2,q) respectively, and the fact that p and q are conjugated, we finally obtain, for $p \neq 2$,

$$\begin{cases} U_{p}(X_{n}) & \leq M(n, m, p) \\ M(n, m, p) = \log(2\pi) + \frac{2}{n(2-p)} \left((p-1) \log \Gamma\left(\frac{n}{2}\right) + \log \Gamma\left(\frac{(n+m)p-2n}{4}\right) \\ -\log \Gamma\left(\frac{(n+m)p}{4}\right) + p \log \Gamma\left(\frac{n+m}{4}\right) \\ +(2-p) \log \Gamma\left(\frac{n(p-2)+mp}{4p}\right) - 2 \log \Gamma\left(\frac{2n(p-2)+(n+m)p}{4p}\right) \\ -\log \Gamma\left(\frac{n(p-2)+mp}{2p}\right) + \log \Gamma\left(\frac{2n(p-2)+(n+m)p}{2p}\right) \\ + \frac{2}{n} \log \Gamma\left(\frac{n}{2q}\right) + \frac{2(q-1)}{n(2-q)} \log \Gamma\left(\frac{n}{q}\right) \end{cases}$$

$$(25)$$

Now using the asymptotics of the log-gamma function [30, 6.1.41] and tedious algebra, one obtains that the upperbound M verifies

$$M(n, m, p) = \log(2\pi) + \frac{\log p}{p - 2} + \frac{\log q}{q - 2} + o(1)$$
(26)

whatever m > 0, dependent or not on n. In the case p = 2, one has by continuity (both for U_p and for M(n, m, p))

$$U_2(X_n) \le M(n, m, 2) \tag{27}$$

with

$$\begin{split} M(n,m,2) &= \log(2\pi) + \tfrac{2}{n} \left(\log \Gamma\left(\tfrac{n}{4}\right) - \log \Gamma\left(\tfrac{n}{2}\right) + \log \Gamma\left(\tfrac{m}{4}\right) - \log \Gamma\left(\tfrac{n+m}{4}\right) \right) \\ &+ \tfrac{1}{2} \, \psi\left(\tfrac{n}{2}\right) - \tfrac{m}{2\,n} \, \psi\left(\tfrac{m}{2}\right) + \tfrac{m-n}{2\,n} \, \psi\left(\tfrac{m+n}{2}\right) + \psi\left(\tfrac{m+n}{4}\right) \end{split}$$

Using again the asymptotics of the log-gamma and psi functions, one obtains

$$M(n, m, 2) = 1 + \log \pi + o(1) \tag{28}$$

Together with (6) these results confirm (21): the lower bound of (6) is attained when $n \to \infty$ in the Student-t case.

3.2.1 The Student-t case with m = n + 2 degrees of freedom: a completely analytical study

Assuming m = n + 2, the Bessel function in (17) is of order $-\frac{1}{2}$ and from [29, 8.469–3] we have,

$$K_{-\frac{1}{2}}(r) = K_{\frac{1}{2}}(r) = \sqrt{\frac{\pi}{2r}} e^{-r}$$
 (29)

Notice first that from the analogy with (7)-(11) and using the Euler's duplication formula [29, 8.335–1], \widetilde{X}_n is distributed according to

$$\widetilde{f}_n(x) = \frac{2^{n-1} \Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}} \Gamma(n)} \exp\left(-2 \left(x^t x\right)^{\frac{1}{2}}\right). \tag{30}$$

This distribution is called the multivariate generalization of the Laplace distribution ³ [36] or multivariate extension generalized Gaussian [37] or multivariate exponential power [35]. As shown in [38], each particle of an ideal relativistic photon gas in a container with rigid and diathermic walls has a 3-dimensional momentum X that follows distribution (30).

Theorem 6 The entropy sum in the case of the Student-t distribution with m = n + 2 degrees of freedom is, for $p \neq 2$,

$$U_p(X_n) = \log \pi + \frac{2\log q - q\log 2}{q - 2} + \frac{2}{n(2 - p)} \left(\log 2 + \log \Gamma(n) - \log \Gamma\left(\frac{n}{2}\right) + \log \Gamma\left(\frac{(n+1)p-n}{2}\right) - \log \Gamma\left(\frac{(n+1)p}{2}\right)\right)$$
(31)

and, for p = 2,

$$U_2(X_n) = 1 + \log\left(\frac{\pi}{2}\right) + \frac{n+1}{n}\left(\psi(n) - \psi\left(\frac{n}{2}\right) - \frac{1}{n}\right)$$
(32)

Moreover, it holds asymptotically for $p \neq 2$

$$\frac{H_{\frac{p}{2}}(X_n) + H_{\frac{q}{2}}(\widetilde{X}_n)}{n} = \log(2\pi) + \frac{\log p}{p-2} + \frac{\log q}{q-2} + o(1)$$
(33)

³ See however [33,34,35] for a different definition of generalized multivariate Laplace distributions.

while for p = 2,

$$\frac{H(X_n) + H(\widetilde{X}_n)}{n} = 1 + \log \pi + o(1) \tag{34}$$

Proof. (31) can be easily deduced from (18) using the fact that $q = \frac{p}{p-1}$ and Euler's duplication formula [29, 8.335–1]. The limit when $p \to 2$ can be obtained from (19) or using a first order expansion of (31) and formula [29, 8.365–1]. This last result can also be found starting directly from Shannon entropy. Using [30, 6.1.41 and 6.3.18], (33) and (34) follow straightforwardly.

These results show that the Student-t distribution with m = n+2 (or, by conjugation, the multivariate exponential power pdf (30)) achieves asymptotically equality in (6) (and (2)). Contrarily to the result of Abe [2] for the Cauchy distribution and Shannon entropy, the results obtained here are in analytical form. We will discuss in section 3.4 about the possible explanations of this asymptotic behavior.

3.2.2 The Student-t case with 1 degree of freedom: revisiting the Cauchy case

We deal in this section with the case of a multivariate Cauchy distribution as previously studied by Abe in [2], *i.e.*

$$f_n(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} (1 + x^t x)^{-\frac{n+1}{2}}$$
(35)

Formulas (18) and (19) apply with m = 1: unfortunately, the remaining integrals in this case cannot be further simplified and numerical evaluations are required. Notice that in the Shannon case, the result found by Abe in [2] is recovered, except that one of the two integrals, numerically evaluated in [2], is in fact analytically expressed here.

Curves in Fig. 1 depict the behavior of (19) and (18) as a function of n, in the Cauchy case, (the integral is numerically evaluated), for p=2 (Shannon case), p=3 and p=10 respectively. The same behavior is observed for any value of p tested. The first curve simply confirms the results by Abe [2]. It also appears from these curves that the conclusion found by Abe remains true for the Rényi version of the uncertainty relation, whatever the possible values of p and thus confirm (21). Hence, this behavior is not specific to the Shannon entropy.

3.3 The general Student-r case

We consider now the case where X_n is distributed according to a Student-r law with m degree of freedom⁴, *i.e.*

$$f_n(x) = \frac{\Gamma\left(\frac{m}{2} + 1\right)}{\pi^{\frac{n}{2}}\Gamma\left(\frac{m-n}{2} + 1\right)} \left(1 - x^t x\right)_+^{\frac{m-n}{2}}$$
(36)

where m > n-2 and $(.)_{+} = \max(.,0)$. When m = n, vector X_n is uniformly distributed in the *n*-dimensional sphere $x^t x = 1$. The Student-r variables play also an important role in probability, since they appear as n-dimensional marginals of the uniform distribution on the sphere in \mathbb{R}^{m+2} ; they are also maximizers (for $\lambda = 1 + \frac{2}{m-n}$) of the Rényi/Tsallis entropy under covariance constraint [12,21,22,31].

Theorem 7 If X_n is Student-r distributed, then the pdf of $||X_n||$ is

$$D_n(r) = \frac{2\Gamma\left(\frac{m}{2} + 1\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m-n}{2} + 1\right)} r^{n-1} (1 - r^2)^{\frac{m-n}{2}}$$
(37)

for $r \in [0;1]$ and 0 otherwise, while the pdf of $\|\widetilde{X}_n\|$ is

$$E_n(r) = \frac{2^{\frac{m-n}{2}+1}\Gamma\left(\frac{m}{2}+1\right)\Gamma^2\left(\frac{m-n}{4}+1\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m-n}{2}+1\right)} r^{-\frac{m-n}{2}-1} J_{\frac{m+n}{4}}^2(r)$$
(38)

for $r \in [0; +\infty)$ and 0 otherwise.

Moreover, if $q > \frac{4n}{m+n+2}$, the sum of the entropy rates is

$$U_{p}(X_{n}) = \log \pi + \frac{(4 + (m - n) q) \log 2}{2 n (2 - q)}$$

$$+ \frac{2}{n (2 - p)} \left((p - 1) \log \Gamma \left(\frac{n}{2} \right) + \log \Gamma \left(\frac{(m - n) p}{4} + 1 \right)$$

$$- \log \Gamma \left(\frac{(m - n) p + 2 n}{4} + 1 \right) - p \log \Gamma \left(\frac{m - n}{4} + 1 \right) \right)$$

$$+ \frac{2}{n (2 - q)} \log \int_{0}^{+\infty} r^{\frac{-(m + n) q}{4} + n - 1} \left| J_{\frac{m + n}{4}}(r) \right|^{q} dr$$

$$(39)$$

 $[\]overline{^4}$ m is named degree of freedom by misuse of language and may be called shape parameter. This choice is adopted by analogy with the Student-t case.

for any m > n - 2, while, for p = 2,

$$U_{2}(X_{n}) = \log(2\pi) + \frac{2}{n} \left(\log \Gamma \left(\frac{m-n}{2} + 1 \right) - \log \Gamma \left(\frac{m}{2} + 1 \right) \right)$$

$$- \log \Gamma \left(\frac{m-n}{4} + 1 \right) + \frac{m+n}{4n} \left(\psi \left(\frac{n}{2} \right) + 2 \psi \left(\frac{m-n}{4} + 1 \right) \right)$$

$$- \frac{m}{n} \psi \left(\frac{m-n}{2} + 1 \right) + \frac{3m-n}{4n} \psi \left(\frac{m}{2} + 1 \right)$$

$$- \frac{2^{\frac{m-n}{2}+1} \Gamma \left(\frac{m}{2} + 1 \right) \Gamma^{2} \left(\frac{m-n}{4} + 1 \right)}{n \Gamma \left(\frac{n}{2} \right) \Gamma \left(\frac{m-n}{2} + 1 \right)} \int_{0}^{+\infty} r^{-\frac{m-n}{2}-1} J_{\frac{m+n}{4}}^{2}(r) \log J_{\frac{m+n}{4}}^{2}(r) dr$$

$$(40)$$

Proof. (37) is directly issued from (36) while (38) is obtained from (12) and [28, 9-28 (3)] (or [27, 8.5 (33)] or [29, 6.567–1]). Then, plugging expressions (37) and (38) into (8) and using [29, 8.380–1] yields (39) for $p \neq 2$ and for any m > n - 2.

In the Shannon case, we can again start from (9). Using the same technique as in section 3.2 and result [29, 8.380–1 and 6.574–2], we obtain (40).

From the asymptotics [30, 9.1.7 and 9.2.1] of the Bessel function for small and large argument, the integral converges provided that

$$q > \frac{4n}{m+n+2} \tag{41}$$

Again, in the general case, neither (39) nor (40) can be further analytically developed and recourse to numerical integration for the remaining integral is needed. Note the symmetry between (39)-(40) and (18)-(19) respectively that can be explained by remarking the symmetry between the Student-t and Student-r variables as evoked in [37].

3.3.1 The Student-r case with m = n degrees of freedom: the uniform case

In this case, we directly apply formulas (39) and (40), where the remaining integrals are numerically evaluated. Figures 2 depicts then the behavior of (40) and (39) as a function of n, for q=2.1 (near the Shannon case), q=3 and q=10 respectively. The same behavior is observed for any value of p tested, showing again that the lower bounds of the uncertainty relations are attained as n increases. Again, we currently try to end analytically the investigation. Note that we observe an increasingly slower convergence as q approaches 2 (especially for "small" m, e.g. uniform): we choose to present the case q=2.1 since the convergence is not too slow compared to other values of q.

3.3.2 Some other Student-r cases

Again we do not enter into details in this subsection. However, other cases were studied as illustrated in Fgure 3 for m=n+2 and m=2n and parameters $q=2,\ q=3$ and q=10 respectively, showing that the asymptotic behavior holds. Contrary to the Student-t case, in the Student-r case we do not find any specific case where a completely analytical study can be performed although $J_{k+\frac{1}{2}}$ admits explicit formulation when $k\in\mathbb{N}$ [29, 8.462].

3.4 Discussion

The preceding results show that any Student-t or Student-r vectors reach asymptotically the case of equality in the Bialynicki-Birula & Mycielski inequality or in its Rényi extension. As the only exact (finite dimensional) case of equality is met by Gaussian vectors, one may be tempted to explain this asymptotic behavior by a "Gaussianization effect" of these vectors, namely the fact that they become "more and more Gaussian" in some sense as n increases. In the rest of this paragraph, we study two possible measures of Gaussianization: in the distribution sense and the information divergence sense.

3.4.1 Convergence in distribution

A well-known property of a Student-t vector X_n is that it can be expressed as a Gaussian scale mixture [25], namely ⁵

$$X_n \stackrel{\mathrm{d}}{=} \sqrt{A_m} \times G_n,\tag{42}$$

where A_m is a scalar inverse Gamma random variable 6 inv $\Gamma\left(\frac{m}{2},2\right)$ with shape parameter m/2 and scale parameter 1/2, independent of the zero-mean Gaussian vector G_n with identity covariance matrix (see also [31,37,39]). In the particular Cauchy case m=1, one recovers the fact that A_1 is a Lévy variable [40]. Since $E\left[\sqrt{m-2}\sqrt{A_m}\right] = \sqrt{\frac{m-2}{2}} \frac{\Gamma(\frac{m-1}{2})}{\Gamma(\frac{m}{2})} = 1 + O(1/m)$ and $VAR\left[\sqrt{m-2}\sqrt{A_m}\right] = 1 - \frac{m-2}{2} \left(\frac{\Gamma(\frac{m-1}{2})}{\Gamma(\frac{m}{2})}\right)^2 = \frac{1}{2m} + o(1/m)$, we deduce that $\sqrt{m-2}\sqrt{A_m}$ converges to 1 almost surely when m goes to infinity. Hence, by Slutsky's theorem, when m goes to infinity with n, any subvector X_n^k of X_n

 $[\]frac{1}{5} \stackrel{d}{=} \text{means equality in distribution.}$

⁶ The inverse Gamma distribution $\operatorname{inv}\Gamma(\alpha,\beta)$ writes $f(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}x^{-1-\alpha}\exp\left(-\frac{\beta}{x}\right)$ it is the distribution of the inverse of a $\Gamma(\alpha,1/\beta)$ random variable.

with finite dimension k < n converges in distribution to a Gaussian vector of finite size k [41].

However, when m remains constant and n grows, since variable A_m does not depend on the dimension n, any subvector X_n^k of X_n of finite dimension k < n remains Student-t with m degree of freedom: as $n \to \infty$, no Gaussianization effect happens for constant m, at least in the distribution sense. As an example, this is the case for a Cauchy vector for which it is well-known that any subvector X_n^k remains Cauchy distributed whatever n.

For the Student-r random vectors, the scale mixture representation does not hold. However, it is shown in [37] that any Student-r vector X_n can be expressed as

$$X_n \stackrel{\mathrm{d}}{=} \frac{G_n}{(\|G_n\|^2 + B_{m,n})^{\frac{1}{2}}} \tag{43}$$

where $B_{m,n}$ is a scalar Gamma distributed variable 7 , with shape parameter $\alpha = (m-n+2)/2$ and scale parameter $\beta = 2$, which is independent on the unit covariance Gaussian vector G_n . This representation was given in [37] for m integer, but it holds for non integer values of m as well. Now, variable $C_m = \|G_n\|^2 + B_{m,n}$ is Gamma distributed $\Gamma\left(\frac{m+2}{2},2\right)$ and then, with the same technique as in the Student-t case, $\sqrt{\frac{m+2}{C_m}} \to 1$ almost surely when $n \to \infty$ (since m > n-2, $m+2 \to \infty$ when $n \to \infty$). Although C_m and G_n are not independent, one can again evoke Slutsky's theorem [41] to conclude that a finite-size subvector of X_n^k of X_n tends in distribution to a Gaussian vector when n tends to infinity. This result is known as "Poincaré's observation" and gave birth to an important literature; despite its name, it is attributed to Borel in [42] and to Mehler in [43]. It is illustrated in figure 4 in the uniform case for k=1 and various values of n, and for k=2 and n=10.

3.4.2 Convergence in the Kullback-Leibler divergence rate sense

The Kullback-Leibler (KL) divergence between a random vector Y_n and a random vector Z_n with respective pdfs ρ_y and ρ_z is defined as

$$D_{kl}(Y_n \parallel Z_n) = D_{kl}(\rho_y \parallel \rho_z) = \int \rho_y \log \left(\frac{\rho_y}{\rho_z}\right)$$
(44)

and is a measure of similarity between these two random vectors. This divergence is nonnegative and is zero if and only if $\rho_z = \rho_y$ (i.e. $Z_n \stackrel{d}{=} Y_n$) [13]. This

⁷ The Gamma distribution $\Gamma(\alpha, \beta)$ writes $f(x) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)}x^{\alpha-1}\exp\left(-\frac{x}{\beta}\right)$.

divergence has also a physical signification, as shown e.g. in [44,45] Note that KL divergence is not symmetric. In the elliptical framework, its expression can be simplified using the following lemma.

Lemma 8 If Y_n and Z_n are elliptical with the same characteristic matrix $\Sigma_n = I_n$, then

$$D_{kl}(Y_n \| Z_n) = D_{kl}(\|Y_n\| \| \|Z_n\|)$$
(45)

Proof. By definition,

$$D_{kl}(Y_n \parallel Z_n) = \int_{\mathbb{R}^n} \rho_y(x) \log \left(\frac{\rho_y(x)}{\rho_z(x)}\right) dx$$
$$= \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^{+\infty} r^{n-1} d_y(r) \log \left(\frac{d_y(r)}{d_z(r)}\right) dr$$

with $\rho_Y(x) = |d_Y\left((x^t x)^{\frac{1}{2}}\right)|$ and using [29, 4.642]. But using the expression of $\rho_{\|Y_n\|}(r)$ as given by (11), we deduce

$$D_{kl}(Y_n \parallel Z_n) = \int_{0}^{+\infty} \rho_{\parallel Y_n \parallel}(r) \log \left(\frac{\rho_{\parallel Y_n \parallel}(r)}{\rho_{\parallel Z_n \parallel}(r)} \right) dr = D_{kl}(\parallel Y_n \parallel \parallel \parallel Z_n \parallel)$$

■ For $\Sigma_n \neq I_n$, Y_n and Z_n must be replaced by $\Sigma^{-\frac{1}{2}}Y_n$ and $\Sigma^{-\frac{1}{2}}Z_n$ respectively.

Applying this result to the Student-t vector X_n under study and a zero-mean Gaussian vector G_n with the same covariance matrix yields the following results.

Theorem 9 For m > 2, the KL divergences between a Student-t vector and a Gaussian vectors with the same covariance matrix are given by

$$D_{kl}(X_n \parallel G_n) = \frac{n}{2} \log \left(\frac{2e}{m-2} \right) + \log \Gamma \left(\frac{n+m}{2} \right) - \log \Gamma \left(\frac{m}{2} \right) + \frac{m+n}{2} \left(\psi \left(\frac{m}{2} \right) - \psi \left(\frac{n+m}{2} \right) \right)$$

$$(46)$$

and

$$D_{kl}(G_n \parallel X_n) = -\frac{n}{2} \log \left(\frac{2e}{m-2}\right) - \log \Gamma\left(\frac{n+m}{2}\right) + \log \Gamma\left(\frac{m}{2}\right) + \frac{m+n}{2}J(n)$$

$$(47)$$

where

$$J(n) = \frac{\int_{0}^{+\infty} r^{n-1} \log(1+r^2) e^{-\frac{m-2}{2}r^2}}{2^{\frac{n}{2}-1} (m-2)^{-\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)}$$
(48)

Moreover⁸ when $n \to \infty$ and $m \to \infty$ with n,

$$D_{\rm kl}(X_n \parallel G_n) \sim \frac{n}{2} \log \left(\frac{m}{m-2}\right) + \frac{1}{2} \log \left(\frac{m}{n+m}\right) - \frac{n}{2m} - \frac{n(n+2m)}{6m^2(n+m)}$$
 (49)

whereas, when m = O(1) we have

$$D_{kl}(X_n \parallel G_n) = \frac{n}{2} \left(\log \left(\frac{2}{m-2} \right) + \psi \left(\frac{m}{2} \right) \right) + o(n)$$
 (50)

and bounds for function J(n) are given by

$$J(n) \ge \psi\left(\frac{n}{2}\right) - \log\left(\frac{m-2}{2}\right) + \log\left(1 + \frac{m-2}{2}\left(\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}\right)^2\right) \tag{51}$$

$$J(n) \le \log\left(1 + \frac{n}{m}\right) + \frac{2n}{(n+m)(m-2)} \tag{52}$$

Proof. Note first that if m > 2 the covariance matrix of X_n with pdf (15) is $\frac{1}{m-2}I_n$ [37]; if $m \le 2$, this covariance does not exist. Then starting from (45), using (16) for X_n and (7)-(11) for G_n (with covariance $\frac{1}{m-2}I_n$), the KL divergence between X_n and G_n writes

$$D_{kl}(X_n \parallel G_n) = \frac{n}{2} \log 2 + \log \Gamma\left(\frac{n+m}{2}\right) - \log \Gamma\left(\frac{m}{2}\right) - \frac{n}{2} \log(m-2)$$

$$+ \frac{m-2}{2} \int_{0}^{+\infty} r^2 D_n(r) dr - \frac{m+n}{2} \frac{2\Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)} \int_{0}^{+\infty} \frac{r^{n-1}}{(1+r^2)^{\frac{n+m}{2}}} \log(1+r^2) dr$$

The first integral term is equal to $\frac{n}{m-2}$ while the second one is evaluated noticing that $\int h^k(r) \log(h(r)) dr = \frac{\partial}{\partial \lambda} \int h^{\lambda}(r) \Big|_{\lambda=k}$ and with [29, 8.380–3], leading to (46).

The KL divergence between G_n and X_n , (47) is derived from the same technique.

 $[\]overline{{}^8} \ f \sim g$ means that $f/g \to 1$ or f = g + o(g)

Using the asymptotics [30, 6.1.41 and 6.3.18] of the log-gamma and of the psi functions up to the second order term, with tedious algebra, (49) follows.

One can easily check that for any value a and for $r \geq 0$ we have

$$\log(1+r^2) \le \log(1+a^2) + \frac{r^2 - a^2}{1+a^2}$$

Using this inequality with $a^2 = n/m$ in J(n) leads to (52). Likewise, $\log(1 + r^2) = 2\log r + \log(1 + r^{-2})$ and for any value a and for $r \ge 0$ we have

$$\log(1+r^{-2}) \ge \log(1+a^{-2}) - \frac{2}{a(1+a^2)}(r-a)$$

Hence, plugging this inequality into J(n) and using [29, 4.352–1] leads to

$$J(n) \ge \psi\left(\frac{n}{2}\right) - \log\left(\frac{m-2}{2}\right) + \log(1 + a^{-2}) - \frac{2}{a(1+a^2)} \left(\sqrt{\frac{2}{m-2}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} - a\right)$$

The best bound is obtained by maximizing the right-hand side, which amounts to choose $a = \sqrt{\frac{2}{m-2}} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}$ leading to (51).

Results from the above theorem show that D_{kl} does not generally tend to zero with n. However, as previously noted, it is more significant to study the KL divergence rates $\frac{1}{n}D_{kl}$. Three situations occur:

- If n = o(m), (49) shows that $D_{kl}(X_n \parallel G_n)$ tends to 0. Thus $D_{kl}(X_n \parallel G_n)/n$ tends also to 0 when n increases. This behavior shows that both in the KL divergence sense and in the KL divergence rate sense, a Gaussianization effect happens. Using the asymptotics [30, 6.1.41 and 6.3.18], we obtain that $D_{kl}(G_n \parallel X_n) = \frac{n+m}{2}(J(n) \log(1+n/m)) + \varepsilon(n)$ where $\varepsilon(n) = o(1)$. Using (52) leads to $D_{kl}(G_n \parallel X_n) \leq \frac{n}{m-2} + \varepsilon(n)$. Together with the positivity of the KL divergence, $D_{kl}(G_n \parallel X_n)$ tends to 0 when $n \to \infty$: this confirms the conclusion drawn from $D_{kl}(X_n \parallel G_n)$.
- If $m \to \infty$ in m = O(n), (49) tells us that the KL divergence $D_{kl}(X_n \parallel G_n)$ has a finite non-zero limit or can even diverge (e.g. if m = o(n)): again the rate $\frac{1}{n}D_{kl}(X_n \parallel G_n)$ tends to 0 with $n \to \infty$. In this situation, there is no Gaussianization in the KL divergence sense, but in divergence rate the Gaussianization effect remains. From (51), the same technique as in the first case shows that the lower bound of $D_{kl}(G_n \parallel X_n)$ tends to a non-zero limit (and clearly diverges if m = o(n)). From the upperbound (52), it appears that $\frac{1}{n}D_{kl}(G_n \parallel X_n)$ tends to 0 and the conclusion drawn from $D_{kl}(X_n \parallel G_n)$ holds.
- If m = O(1) (e.g. m constant),

- · from (50) one can conclude that no Gaussianization appears, neither in KL divergence, nor in KL divergence rate.
- · from the lower bound (51), one can check that $D_{\rm kl}(G_n \parallel X_n)$ diverges with n. However, from (52) it appears that $\frac{1}{n}D_{\rm kl}(G_n \parallel X_n)$ tends to 0. This behavior seems contradictory with the previous one and tells that in fact a Gaussianization effect exists in KL divergence rate. This contradiction is possible since the KL divergence in not symmetric. Note also that when $m \leq 2$ (47)-(51)-(52) can still be considered without the normalization m-2 (no covariance for X_n): the conclusion holds in this case.

The KL divergence rate $\frac{1}{n}D_{kl}(X_n||G_n)$ is in concordance with the observation of the Gaussianization effect in the distribution sense. This generally holds for $\frac{1}{n}D_{kl}(G_n||X_n)$, except notably when m does not go to infinity: in this case, although there is no Gaussianization in distribution the KL divergence rate goes to 0 with n. Furthermore, when m goes to infinity, although X_n reaches asymptotically the bound in the B.B.M.I., its distribution stays generally at infinite – or at least at non zero distance – "distance" (in the KL divergence sense) from the Gaussian distribution. Thus, much care must be taken with these conclusions, especially about the real meaning of the KL divergence rate.

Theorem 10 For any m > n - 2, the KL divergences between a Student-r vector and a Gaussian vector with the same covariance matrix is

$$D_{kl}(X_n \parallel G_n) = \frac{n}{2} \log \left(\frac{2e}{m+2}\right) + \log \Gamma\left(\frac{m}{2} + 1\right) - \log \Gamma\left(\frac{m-n}{2} + 1\right) + \frac{m-n}{2} \left(\psi\left(\frac{m-n}{2} + 1\right) - \psi\left(\frac{m}{2} + 1\right)\right)$$

$$(53)$$

Moreover it verifies the asymptotic

$$D_{\rm kl}(X_n \parallel G_n) \sim \frac{1}{2} \log \left(\frac{m+2}{m-n+2} \right) - \frac{n(m-n)}{(m+2)(m-n+2)}$$
 (54)

Proof. Note first that the covariance matrix of X_n as defined by (36) is $\frac{1}{m+2}I_n$ [37]. Then, (53) is obtained in the same way as (46).

Notice moreover that here, since $m \ge n$, m goes to infinity with n. Using , as in the Student-t case, the asymptotic expansions [30, 6.1.41 and 6.3.18] of the log-gamma and of the psi functions up to the second order term, (54) follows.

Thus, in the Student-r case, two behaviors arise:

• If n = o(m), from (54) $D_{kl}(X_n \parallel G_n)$ tends to zero n, hence the KL divergence rate goes also to 0: both in the KL divergence and in the KL divergence rate sense a Gaussianization effect appears.

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• If n = O(m), the KL divergence does not tends to zero and can even diverge with n if m - n = o(n) (e.g. in the uniform case). But $D_{kl}(X_n \parallel G_n)/n$ still tends to 0 with n, exhibiting again an asymptotic Gaussianization behavior.

These observations can be linked to a famous result by Diaconis and Freedman [42] that quantifies the total variation divergence 9 between X_n and G_n as

$$D_{\text{tv}}(X_n, G_n) \le \frac{2(n+3)}{m-n-1} \quad \forall \ 1 \le n \le m-2.$$
 (55)

for integer m, where X_n is built from the n first components of a (m + 2)-dimensional vector uniformly distributed in the surface of the (m + 2)-dimensional sphere [42]. Moreover, a necessary and sufficient condition for convergence to 0 of $D_{\text{tv}}(X_n, G_n)$ is n = o(m). These results were extended to the KL divergence by O. Johnson [43] as follows:

$$D_{kl}(X_n \parallel G_n) \le \log\left(\frac{m}{m-n}\right) + \frac{2}{\sqrt{m+2}/C - 1}$$
 (56)

A converse result is also provided in [43], stating that if $D_{kl}(X_n \parallel G_n) \to 0$ then n = o(m).

4 Concluding remarks and discussion

In this paper, we first extended the entropic uncertainty relation found by Bialynicki-Birula & Mycielski to Rényi entropies. We have checked that, for a given dimension n, the bound is again attained in the Gaussian case and in this case only. We analytically proved that the bound is also asymptotically attained with the dimension n in the conjugate multivariate exponential power case, whatever $p \geq 1$. We numerically showed that the bound is asymptotically attained for the Cauchy case for any value of p > 2, extending the results of Abe, as well as in the general Student-t case, including these two cases. This asymptotic analysis was confirmed analytically. The same conclusion was drawn in the Student-r context, as far as our numerical simulations are valid. These results seem to violate the fact that the bound is attained *only* for the Gaussian case. If a Gaussianization effect was evoked in the first example, the second one showed that the effect of dimension only may be suspected, provided some favorable conditions exist. To get a feeling to this interpretation,

We recall that the total variation divergence is the \mathcal{L}_1 -norm difference $D_{\text{tv}}(Y, Z) = \int_{\mathbb{R}^n} \|\rho_Y - \rho_Z\|$

let us come back to the Beckner relation (4), and assume that we deal with a wave function Ψ_n such that

$$(C_{p,q})^n \frac{\|\Psi_n\|_p}{\|\widehat{\Psi}_n\|_q} = h(n,p) + o(h(n,p))$$
 (57)

for large n and for some function h(n, p). Again by taking the logarithm and using the same approach as in section 2, it is easy to show that

$$\frac{H_{\frac{p}{2}}(X_n) + H_{\frac{q}{2}}(\widetilde{X}_n)}{n} = \log(2\pi) + \frac{\log p}{p-2} + \frac{\log q}{q-2} + \frac{2p\log(h(n,p))}{n(2-p)} + o(1)$$

As a conclusion, it is sufficient that $\frac{\log(h(n,p))}{n}$ tends to 0 as n increases: h(n,p) does not need to converge to 1, showing that no Gaussianization effect is needed here; in fact, h(n,p) can even diverge with n. This conclusion holds in the Shannon case, provided that $\frac{h(n,p)}{p-2}$ (and the remaining term) has a limit as $p \to 2$. In fact, in the conjugate exponential power case (Student-t with m=n+2), function h(n,p) writes $h(n,p)=h(p)=\frac{\log p}{2p}-\frac{\log 2}{2q}$, illustrating this conclusion. As perspective, the Student-r case should be analytically solved to confirm the numerical investigations. To go further, it seems interesting to determine what are the minimal conditions a pdf f_n should verify so that it attains asymptotically the bound in the B.B.M.I. As far as we know, this question remains open and we still investigate it. We suspect however that in the elliptical context, no major additional constraints are needed to reach the same conclusion. Indeed, we feel that in this framework the normalization term 1/n in the entropy rates may be strong: one can invoke Poincaré's observation inducing a "Gaussianization" and we suspect that the remaining contribution of the sum of the entropy can diverge, but at most in o(n).

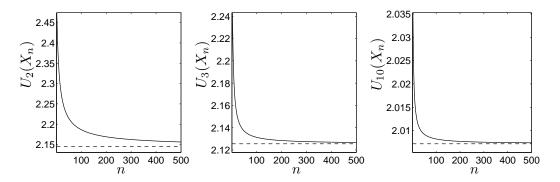


Fig. 1. Illustration of the uncertainty relation (6) in the Cauchy context, for p = 2 (Shannon version (2)), p = 3 and for p = 10 respectively. The solid line depicts the sum of the entropy rates $U_p(X_n)$ as a function of n, and the dashed line represents the lower bound.

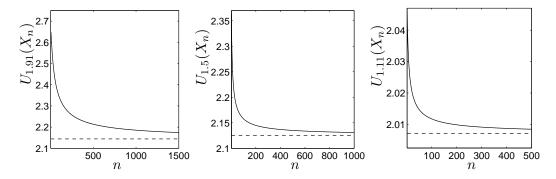


Fig. 2. Illustration of the uncertainty relation (6) in the uniform case, for q = 2.1 (near the Shannon version (2)), q = 3 and for q = 10 respectively. The solid line depicts $U_p(X_n)$ as a function of n and the dashed line depicts the lower bound.

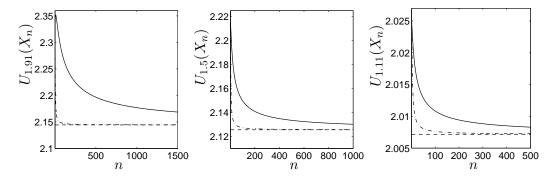
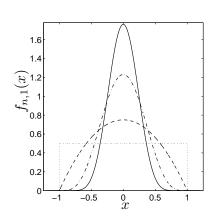


Fig. 3. Illustration of the uncertainty relation (6) in several Student-r contexts, for q=2.1 (near the Shannon version (2)), q=3 and for q=10 respectively. The figures represent the cases m=n+2 (solid line) and m=2n (dashed-dotted line). The small dashed line represents the lower bound.

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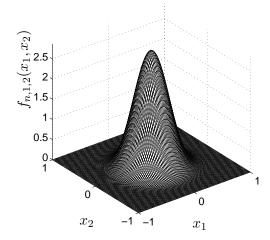


Fig. 4. Illustration of the convergence in distribution of the a finite size uniformly distributed vector to the Gaussian pdf: pdf $f_{n,1}(x)$ of the first component and for several values of n (left) and pdf $f_{n,1,2}(x_1, x_2)$ of the first two components for n = 10 (right). The dotted line depicts n = 1, the dashed line represent n = 2, the dashed line depicts n = 5 and the solid line represents the case n = 10.

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